SINGULAR MEASURES AND l^p FOURIER TRANSFORMS

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ABSTRACT

Let G be the Cantor group or the circle group, and let Γ be the dual of G. There exists a probability measure λ on G, singular with respect to Haar measure on G, such that for $1 \leqq p < 2$ the inequality

$$\left(\int_G |f|^2 d\lambda\right)^{1/2} \leqq K_p\left(\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)|^p\right)^{1/p}$$

holds for trigonometric polynomials f.

1. Introduction

Suppose G and Γ are dual LCA groups. The Hausdorff–Young inequalities show that if $1 \leqq p \leqq 2$ and $f \in L^p(\Gamma)$, then the Fourier transform $\hat{f}$ is defined on G up to sets of Haar measure 0. But in general there is no definition of $\hat{f}$ on $E \subseteq G$ if E is null. In certain cases, though, there exist measures λ on G which are singular with respect to Haar measure and for which inequalities such as

$$\left(\int_G |\hat{f}|^q d\lambda\right)^{1/q} \leqq K \|f\|_{L^p(\Gamma)} \tag{1}$$

hold for certain values of q and p $(1 \leqq p < 2)$. (See, e.g., [6], [4].) We are interested in the case $q = 2$, and for $q = 2$ it is easy to see ([5]) that (1) holds if and only if

$$\hat{\lambda} * L^p(\Gamma) \subseteq L^{p'}(\Gamma) \qquad \left(\frac{1}{p}+\frac{1}{p'}=1\right). \tag{2}$$

For example, if $\hat{\lambda} \in L^r(\Gamma)$ where $r^{-1} = 2 - 2/p$, then (2) holds by Young's

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inequality. But then $p < 4/3$, since $r > 2$ if λ is singular. A more interesting example is the theorem of Tomas and Stein [5] that (1) holds for $G = R^n$, $1 \leqq p \leqq 2(n+1)/(n+3)$, $q = 2$, and λ the usual measure on the surface of the unit sphere in R^n. In this note we prove the following theorem.

THEOREM. *If G is the Cantor group D^∞ or the circle group T and if Γ is the dual of G, then there exists a probability measure λ on G, singular with respect to Haar measure, such that for $1 \leqq p < 2$ the inequality*

$$\left(\int_G |f|^2 d\lambda\right)^{1/2} \leqq K_p \left(\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^p\right)^{1/p} \tag{3}$$

holds for, say, trigonometric polynomials f on G.

Our method works as well for several other LCA groups G. Since the extensions are routine but somewhat lengthy, we sacrifice generality for the sake of brevity.

2. A lemma

The following elementary lemma constitutes the "hard" part of our proofs.

LEMMA 1. *Suppose $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. There exists $\delta_0 = \delta_0(p) > 0$ such that the inequality*

$$\left(\frac{1+\delta}{2}|a+b|^2 + \frac{1-\delta}{2}|a-b|^2\right)^{1/2} \leqq (1 + (3\delta)^q)^{1/q}(|a|^p + |b|^p)^{1/p} \tag{4}$$

holds for $0 \leqq \delta < \delta_0$ and all complex numbers a and b.

PROOF. It is enough to suppose that a and b are positive, so that the LHS of (4) becomes $(a^2 + b^2 + 2ab\delta)^{1/2}$. By scaling we can assume that $a = 1$ and $0 < b < 1$. Thus we will actually establish the inequality

$$(1 + b^2 + 2b\delta)^{1/2} \leqq (1 + (3\delta)^q)^{1/q}(1 + b^p)^{1/p}, \qquad 0 < b < 1. \tag{5}$$

If $0 < b \leqq \delta$, then

$$(1 + b^2 + 2b\delta)^{1/2} \leqq 1 + 3b\delta \leqq (1 + (3\delta)^q)^{1/q}(1 + b^p)^{1/p}.$$

We will prove (5) by showing that for $0 \leqq \delta < \delta_0(p)$, the function

$$F_\delta(b) = \frac{1 + b^2 + 2b\delta}{(1 + b^p)^{2/p}}$$

is decreasing for $\delta \leqq b \leqq 1$.

The derivative $F'_\delta(b)$ has the same sign as

$$(1+b^p)^{2/p}(2b+2\delta)-2(1+b^2+2b\delta)(1+b^p)^{(2/p)-1}b^{p-1}$$
$$=2(1+b^p)^{(2/p)-1}(b+\delta-b^{p-1}-\delta b^p).$$

Thus we need only show that

$$b+\delta < b^{p-1}+\delta b^p \qquad (0 \leqq \delta < \delta_0(p), \quad \delta \leqq b \leqq 1),$$

or that

$$\delta \leqq \frac{b^{p-1}-b}{1-b^p} \doteq s(b) \qquad (0 \leqq \delta < \delta_0(p), \quad \delta \leqq b \leqq 1). \tag{6}$$

In the next paragraph we will show that $s'(b)>0$ for $0<b<1$. Therefore (6) will hold if $s(\delta) \geqq \delta$ or, equivalently, $2 \leqq \delta^{p-2}+\delta^p$. This will hold if

$$\delta \leqq 2^{1/(p-2)} = \delta_0(p).$$

We finish the proof of the lemma by showing that $s'(b)>0$ for $b \in (0,1)$. The sign of $s'(b)$ is the same as the sign of

$$(1-b^p)([p-1]b^{p-2}-1)-(b^{p-1}-b)(-pb^{p-1})=$$
$$(p-1)b^{p-2}-(p-1)b^p+b^{2p-2}-1 \doteq t(b).$$

Now $t(1)=0$, so it is enough to show that $t'(b)<0$ for $b \in (0,1)$. But $t'(b)<0$ is equivalent to

$$0<p+(2-p)b^{-2}-2b^{p-2} \doteq u(b). \tag{7}$$

Since $u(1)=0$ and

$$u'(b)=2(2-p)b^{-3}(b^p-1)<0, \qquad b \in (0,1),$$

it follows that (7) holds.

3. The Cantor group

Let $D=\{0,1\}$ with addition modulo 2 be the two-element group. Then $D^\infty = D \times D \times \cdots$ is the Cantor group. For $g=(g_1,g_2,\cdots)\in D^\infty$, define $\gamma_n(g)$ to be $(-1)^{g_n}$, the nth "Rademacher function" on D^∞. Our measure λ will be any Riesz product $\Pi_{n=1}^\infty(1+\delta_n\gamma_n)$, where $0 \leqq \delta_n < 1$, $\Sigma_{n=1}^\infty \delta_n^2=\infty$, and $\Sigma_{n=1}^\infty \delta_n^q<\infty$ whenever $q>2$. It follows from theorem 4.4 of [3] that such a λ is a (continuous) measure singular with respect to the Haar measure of D^∞. Thus we need to show only that if Γ is the dual of D^∞, then

$$\left(\int_{D^\infty} |f|^2 d\lambda\right)^{1/2} \leqq K_p\left(\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)|^p\right)^{1/p} \tag{8}$$

for $1 \leqq p < 2$ and trigonometric polynomials f on D^∞. This inequality follows from an iteration of the following "product lemma".

LEMMA 2. ([2], chapitre III, lemme 1). *Suppose that $1 \leqq r \leqq s < \infty$ and that (X, μ), (Y, η) are two measure spaces. Suppose also that T_1 and T_2 are bounded linear maps of $L^r(X)$ into $L^s(Y)$ having norms A_1 and A_2 respectively. Consider the product map T defined by $Tf = T_1 g T_2 h$ for f on $X \times X$ of the form $f(x_1, x_2) = g(x_1)h(x_2)$. Then T extends to be a bounded linear map of $L^r(X \times X)$ into $L^s(Y \times Y)$ with norm A_1A_2.*

Since the Fourier transformation of functions on Γ into functions on D^∞ is a product transformation and since λ is a product measure, (8) follows from

$$\prod_{n=1}^{\infty} (1 + (3\delta_n)^q)^{1/q} < \infty \qquad (p^{-1} + q^{-1} = 1)$$

and Lemma 1, restated as the inequality

$$\left(\int_D |f|^2(1 + \delta\gamma_0)dg\right)^{1/2} \leqq (1 + (3\delta)^q)^{1/q}(|\hat{f}(0)|^p + |\hat{f}(\gamma_0)|^p)^{1/p}, \qquad 0 \leqq \delta < \delta_0(p).$$

Here γ_0 is the nontrivial character on D, while dg denotes normalized Haar measure on D.

4. The circle group

We take the interval $[0, 1)$ as our model for the circle group T. For $t \in T$, ε_t will denote the unit point mass at t. Our measure λ will be any infinite convolution

$$\mathop{*}_{n=1}^{\infty} \left(\frac{1+\delta_n}{2}\varepsilon_0 + \frac{1-\delta_n}{2}\varepsilon_{2^{-n}}\right),$$

where $0 \leqq \delta_n < 1$, $\Sigma_{n=1}^\infty \delta_n^2 = \infty$, and $\Sigma_{n=1}^\infty \delta_n^q < \infty$ whenever $q > 2$. Let $\Phi : D^\infty \to T$ be the mapping

$$(g_1, g_2, \cdots) \mapsto \sum_{j=1}^{\infty} g_j 2^{-j}.$$

Then the present λ is the image under Φ of the measure described in Section 3. Since Φ also carries the Haar measure of D^∞ onto that of T, λ is a (continuous) measure singular with respect to the Haar measure of T. The inequality

$$\left(\int_T |f|^2 d\lambda\right)^{1/2} \leqq K_p\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p\right)^{1/p} \qquad (1 \leqq p < 2),$$

for trigonometric polynomials f, follows immediately from the inequality

$$\left(\sum_{g_1,\cdots,g_M\in\{0,1\}} \prod_{j=1}^{M} \frac{1+(-1)^{g_j}\delta_j}{2}\left| f\left(\sum_{j=1}^{M} g_j 2^{-j}\right)\right|^2\right)^{1/2} \leqq \left(\prod_{j=1}^{M}[1+(3\delta_j)^q]\right)^{1/q}\left(\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^p\right)^{1/p}, \tag{9}$$

which we will prove for $0 \leqq \delta_j < \delta_0(p)$ and all M such that $2^M \geqq L = \text{length}(f)$.

Since we can assume that f is of the form $f(t) = \sum_{n=0}^{L-1}\hat{f}(n)\exp(2\pi i n t)$, (9) can be rewritten

$$\left(\sum_{g_1,\cdots,g_M\in\{0,1\}} \prod_{j=1}^{M} \frac{1+(-1)^{g_j}\delta_j}{2}\left| \sum_{n=0}^{2M-1} c_n \exp\left(2\pi i n \sum_{j=1}^{M} g_j 2^{-j}\right)\right|^2\right)^{1/2} \leqq \left(\prod_{j=1}^{M}[1+(3\delta_j)^q]\right)^{1/q}\left(\sum_{n=0}^{2M-1}|c_n|^p\right)^{1/p}. \tag{10}$$

For $M = 1$, this is Lemma 1 again. We would like to iterate Lemma 1 to obtain (10). Lemma 2 does not apply here, but the proof of Lemma 2 can be made to work. So suppose (10) is true, and we will establish the analogue of (10) with M replaced by $M+1$. In the following we will write Π_a^{M+1} in place of

$$\prod_{j=a}^{M+1} \frac{1+(-1)^{g_j}\delta_j}{2}$$

to simplify the typography. Now

$$\left(\sum_{g_1,\cdots,g_{M+1}} \prod_1^{M+1} \left| \sum_0^{2M+1-1} c_n \exp\left(2\pi i n \sum_1^{M+1} g_j 2^{-j}\right)\right|^2\right)^{1/2}$$

$$= \left(\sum_{g_2,\cdots,g_{M+1}} \prod_2^{M+1} \sum_{g_1} \frac{1+(-1)^{g_1}\delta_1}{2}\left| \sum_{k=0}^{1}\exp(\pi i k g_1)\sum_{l=0}^{2M-1} c_{k+2l}\exp\left(2\pi i[k + 2l]\sum_2^{M+1} g_j 2^{-j}\right)\right|^2\right)^{1/2}$$

$$\leqq \left(\sum_{g_2,\cdots,g_{M+1}} \prod_2^{M+1} \cdot (1+(3\delta_1)^q)^{2/q}\left[\sum_{k=0}^{1}\left|\sum_{l=0}^{2M-1} c_{k+2l}\exp\left(2\pi i[k + 2l]\sum_2^{M+1} g_j 2^{-j}\right)\right|^p\right]^{2/p}\right)^{1/2}.$$

Here the inequality follows from Lemma 1. Minkowski's inequality shows that this is dominated by

$$(1+(3\delta_1)^q)^{1/q}\left(\sum_{k=0}^{1}\left[\sum_{g_2,\cdots,g_{M+1}}\prod_{2}^{M+1}\left|\sum_{l=0}^{2M-1}c_{k+2l}\exp\left(2\pi i l\sum_{2}^{M+1}g_j2^{-j+1}\right)\right|^2\right]^{p/2}\right)^{1/p}.$$

An application of (10) finishes the induction and so establishes (9).

5. A comment

It is somewhat interesting to look at the dual formulation of (3):

$$\left(\sum_{\gamma\in\Gamma}|\widehat{f\lambda}(\gamma)|^q\right)^{1/q}\leqq K_p\left(\int_G|f|^2d\lambda\right)^{1/2}. \tag{11}$$

One consequence of (11) is that if μ is the restriction of λ to any Borel subset of G having positive λ-measure, then $\hat{\mu}\in l^q(\Gamma)$. Thus, as a corollary to Section 4, we can state the following.

COROLLARY. *Suppose* $1<p<2$, $p^{-1}+q^{-1}=1$, $0\leqq\delta_j<1$, $\Sigma_{j=1}^{\infty}\delta_j^2=\infty$, *and* $\Sigma_{j=1}^{\infty}\delta_j^q<\infty$. *If* λ *is the measure*

$$\mathop{\ast}_{j=1}^{\infty}\left(\frac{1+\delta_j}{2}\varepsilon_0+\frac{1-\delta_j}{2}\varepsilon_{2^{-n}}\right)$$

on T, *then the restriction of* λ *to any set of positive* λ*-measure is a singular measure with Fourier transform in* $l^q(Z)$.

This complements a result obtained in [1], that $|\lambda(n)|\to 0$ as $|n|\to\infty$ if $\delta_j\to 0$.

REFERENCES

1. J. R. Blum and B. Epstein, *On the Fourier transforms of an interesting class of measures*, Israel J. Math. **10**(1971), 302–305.

2. A. Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) **20**(1970), fasc. 2, 335–402.

3. E. Hewitt and H. S. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Camb. Phil. Soc. **62**(1966), 399–420.

4. E. Prestini, *A restriction theorem for space curves*, Proc. Amer. Math. Soc. **70**(1978), 8–10.

5. P. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81**(1975), 477–478.

6. A. Zygmund, *On Fourier coefficients and transforms of functions of two variables*, Studia Math. **50**(1974), 189–201.

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